Relationships Between Visual Static Models and Students' Written Solutions to Fraction Tasks

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The purpose of this study was to deconstruct the relationship between visual static models and students' written solutions to fraction problems using a large sample of students' solutions. Participants in the study included 162 third-grade and 209 fourth-grade students from 17 different classrooms. Students' written responses to open-ended tasks were examined to determine common solutions and errors when using visual static models. The results indicate that (a) common students errors relate to how students interpret the given model or their own model of the situation, and (b) students' flexibility with visual static models is related to successful written solutions. Students with errors generally demonstrated a lack of flexibility in interpreting their own and the given visual static models. Researchers hypothesize that students' exposure to varied mathematical representations influences their ability to flexibly use static visual representations. They recommend that students have a solid understanding of real-world mathematics situations in order to successfully create and interpret visual static models of mathematics.

Key words: *fractions, mathematics education, real-world mathematics, visual perception, visual static models, visualization*

Introduction

An understanding of fractions provides a foundation for success in future learning of mathematics topics, such as ratios, proportions, percentages, decimals, and algebra (National

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Mathematics Advisory Panel, 2008; Council of Chief State School Officers [CCSSO] and National Governors Association [NGA], 2010). Because of the importance of fraction understanding, documents such as *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 2000), *Foundations for Success* (National Mathematics Advisory Panel, 2008), and the Common Core State Standards for Mathematics (CCSSO and NGA, 2010) recommend intense focus on fractions from fourth through eighth grades. However, many students struggle with basic fraction and rational number concepts in these grade levels (Lamon, 2007; Wu, 2005). A lack of visualization skills offers one explanation for students' difficulties with fractions. The visualization of mathematical concepts plays a pivotal role in how well students apply their fraction understanding to novel situations (Arcavi, 2003).

The Common Core State Standards for Mathematics (CCSSO and NGA, 2010) recommend that students "model with mathematics" and "use appropriate tools strategically" (p. 7). When students develop facility with models and tools for thinking, they are able to analyze situations, draw conclusions, and make connections to other domains of mathematics. Additionally, standards established by NCTM emphasize the importance of representing mathematical concepts while problem solving (2000). Sedig and Laing (2006) describe these visual mathematical representations as "graphical representations that encode causal, functional, structural, logical, and semantic properties and relationships of mathematical structures, objects, concepts, problems, patterns, and ideas" (2006, p. 180). Visual static models, as examined in this study, are a specific type of visual mathematical representation that include fixed pictorial images of mathematical concepts. While visual static models provide one method of representing and problem solving with mathematics, those representations that commonly appear on worksheets or tests, may have elements that are unfamiliar to the student or that do not match the student's own mental representation. An unfamiliar visual model may impact how a student interprets a problem. The purpose of this study was to deconstruct the relationship between visual static models and students' written solutions to fraction problems using a large sample of students' solutions. By using this large sample of students' solution models, we hoped to identify patterns and to generate hypotheses about how students employed the models leading to successful or unsuccessful problem solving outcomes. This type of reverse engineered hypothesis, using a large data set of patterns and relationships to generate theory, has the potential to bridge learning (i.e., how students develop and use models) with teaching practices (i.e., how teachers support students' development of fraction models) (Carpenter, Fennema, & Franke, 1996; Hill, Rowan, & Ball, 2005). In this type of hypothesis generation, our inquiry will "not only enable us to systematically specify what we see, but when they take the form of hypotheses or propositions, they suggest how phenomena might possibly be related to each other" (Strauss & Corbin, 1998, p. 102). The following section gives a brief review of the current literature related to the visualization of mathematical representations.

Review of Literature

Visual Representations in Mathematics

NCTM states, "The ways in which mathematical ideas are represented are fundamental to how people can understand and use those ideas" (2000, p. 67). Therefore, as learners develop clear and sophisticated visualizations of mathematical concepts, they will have a deep understanding of those concepts, and develop what Tall and Vinner (1981) refer to as a *concept image*. In this study, we define a *visual static model*, as a still picture that is either printed or drawn on a page to represent mathematical concepts. In this study, we adopt Arcavi's (2003) definition of *mathematical visualization*: the ability to create, use, interpret, and reflect on images in the mind or on paper. Therefore, students use and create visual static models as they develop mathematical visualization skills. These visualizations support meaningful connections with different types of representations and abstract mathematical concepts. Lesh, Post, and Behr (1987) identify five types of mathematical representations: static pictures, manipulative models, written symbols, real-life situations, and spoken language. Understanding a mathematical concept involves a) recognizing the concept among different types of representation, b) flexibly manipulating the concept within a type of representation, and c) translating the concept from one type of representation to another. Static pictures are of particular interest to this study because static models are what students often develop when problem solving, and are often what students see on tests, worksheets, and in textbooks during typical mathematics instruction (Yeh & McTigue, 2009).

Visual representations alleviate cognitive load during problem solving (Clark, Nguyen, & Sweller, 2006) and allow learners to mentally work on one part of the model without having to keep track of the entire model in their minds (Woleck, 2001). For example, many students automatically picture a square divided equally into three parts, two of which are shaded, when they hear or see the symbol, 2/3. This visual model enables learners to maintain the part-whole meaning of the fraction. Findings by van Garderen (2006) also indicate that visualization skills correlate significantly with students' ability to understand mathematics. High-achieving students often display the highest level of spatial visualization. Likewise, low-achieving students benefit from working with given visual static models (Moyer-Packenham, Ulmer, & Anderson, 2012). Visual models provide a scaffold for students as they develop their own visualization skills. But these models can only be useful to students when the students are able to create an accurate model themselves or interpret a given model and use the model effectively for problem solving.

When students interpret and create visual static models, they develop new knowledge that can be applied to other problem solving situations. Researchers emphasize the importance of engaging students in real-world mathematics (Baruk, 1985, Greer, 1993; Verschaffel, De Corte, & Lasure, 1994; Verschaffel, Greer, & De Corte, 2007). Model generation, selection, and interpretation become key factors in students' success in solving mathematical problems (Martin, Svihla, & Petrick Smith, 2012; Moseley & Okamoto, 2008; Ng & Lee, 2009). In his coordination class theory, diSessa (2002) argues that a student's interpretation of a problem situation is connected to his or her readout (i.e. consistently identifying the important information in a problem situation in order to enact a solution strategy). Proficient problem solvers typically develop complex representations (e.g., pictures, diagrams, or tables) to organize and keep track of their solution strategies (Edens & Potter, 2008; Larkin, McDermott, Simon, & Simon, 1980; Whitin & Whitin, 2001). Unfortunately, many students do not automatically utilize visual static models while problem solving or they create a model that does not reflect the mathematical situation. Students require assistance and guidance from teachers and knowledgeable peers as they select, interpret, and create visual models of mathematics (Abrams, 2001; Moyer & Jones, 2004). This research suggests that complex relationships exist among the visual static models teachers use in instruction, the mental models students create for themselves, and students' strategies when using a model for problem solving.

Visualizing Fraction Concepts

The ways that students come to understand fraction concepts and proportional reasoning has been extensively reviewed. For example, research has identified differences in student understanding of fractions based on discrete and continuous quantities (DeWolf, Bassok, & Holyoak, 2013) and examined students' learning pathways as they developed fraction understanding (Martin et al., 2013). As part of the Rational Number Project, Behr, Lesh, Post, and Silver (1983) identified four mathematical sub-constructs of rational numbers—measure, quotient, ratio, and operator (see also Kieren, 1980; Lamon, 2007), and Kieren (1981) identified five faces of mathematical knowledge building related to rational number understanding—mathematical, visual, developmental, constructive, and symbolic.

Moss and Case (1999) suggest that children have two schemas involved in whole number learning: a numerical schema that allows children to learn the fundamentals of counting, and a global quantitative schema that allows children to make global judgments of quantity. When children are about 9-10 years old, they also have two cognitive schemas for fractions: proportional evaluation and splitting (i.e., halving). These cognitive schemes allow children to understand relative proportions and a semi-abstract understanding of basic fractions such as ½ and ¼. However, Lamon's (2007) summary of the current state of research in proportional reasoning suggests that research in this field needs to include a diversification of research approaches and in-depth analyses of children's thinking.

Methods

Research Questions

In this study we examined a large sample of students' solution models to deconstruct the relationship between visual static models and students' written solutions to fraction problems. The overall research question for this study asked: How do visual static models influence students' written solution methods? The following sub-questions guided data collection procedures and analysis:

1. What types of misconceptions do students' written solutions commonly reveal on fractions tasks involving either given or student-created visual static models?
2. What is the relationship between given or student-created visual static models of fractions concepts and students' written solutions on open-ended problems?

Participants and Setting

The students participating in this study were 162 third-grade students (75 males, 87 females) and 209 fourth-grade students (100 males, 109 females) in 17 classrooms. Students' ethnicities, Socio-Economic Status (SES) and English Language Learner (ELL) services were identified by their classroom teachers and the school district. Third-grade students' ethnicities were Caucasian (75.0%), Hispanic (14.1%), Mixed (4.5%), Asian (3.2%), and African American (2.6%). Fourth-grade students' ethnicities were Caucasian (78.4%), Hispanic (14.4%), Mixed (4.6%), Asian, (1.0%), and Pacific Islander (1.0%). About half of the students received free- or reduced lunch and were classified as low-SES (third grade: 42.3%, fourth grade: 53.6%). A small percentage of students received English Language Learner (ELL) services (third grade: 4.5%, fourth grade: 7.7%). The 17 classrooms were in two different school districts in eight different elementary schools in the western United States.

Data Sources & Instruments

The main data source for this analysis was a set of open-ended assessment items following a unit of fraction instruction. These open-ended items came from four different test-item databases (National Assessment of Educational Progress, Massachusetts Comprehensive Assessment System, Utah Test Item Pool Service, and Virginia Standards of Learning) and included visual and numeric representations of fraction concepts. Five mathematics educators reviewed the test items for content validity and the tests were piloted in six school districts prior to the study to determine item difficulties and reliability measures (Moyer-Packenham et al., 2013).

With the aid of university researchers, classroom teachers administered the assessment at the end of their regular unit of fraction instruction. Instructional objectives for these units related directly to state curriculum standards. Third-grade objectives included understanding equal parts; understanding and using region, set, and number line models; naming and writing fractions; comparing and ordering fractions; and understanding equivalent fractions. Fourth-grade objectives included dividing regions into fractional parts; understanding part/whole ideas; comparing and ordering fractions; identifying numbers between fractions; identifying and generating equivalent fractions; modeling addition and subtraction of fractions; and adding and subtracting fractions.

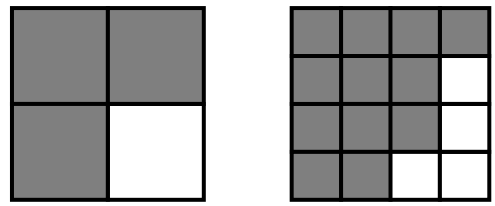
The two open-ended assessment items that form the basis of this analysis highlight students' use of visual static models in their written solutions and were designed to gather information beyond simple correct or incorrect responses (Cai, Lane, & Jakabcsin, 1996). The purpose of these open-ended tasks was to understand the relationships between either given or student-created visual static models of fraction concepts and students' written solutions. The Area Task required third-grade students to interpret models of equivalent fractions. The Pizza Task presented fourth-grade students with a situation of equivalent fractions of different-sized wholes.

Area task. The Area Task (grade 3) assessed students' understanding of equivalent fractions by presenting students with a 2 by 2 square area model of 3/4 and a 4 by 4 square area model of 12/16. In one type of model for the task, the 12 smaller squares in the model of 12/16 occupied adjacent spaces on the larger square (see Figure 1a). A second type of model for the task presented the 12 smaller squares in scattered spaces on the larger square (see Figure 1b). The problem required students to decide if each square had the same fraction of shaded area and explain their thinking with a diagram and words.

Pizza task. The Pizza Task (grade 4) presented one of two similar sharing situations to students, (a) two people each eating half of different pizzas and (b) two people each eating equivalent fractions of different pizzas (4 out of 10 slices and 2 out of 5 slices). In each case, one person (José) claimed to have eaten more pizza than the other person (Ella). The problem then required students to determine how José could be correct (see Figure 2).

In these situations José can be correct if his original pizza was larger than Ella's. In other words, one whole could have been larger than the other whole. These tasks assessed students' understanding of part-whole relationships in fractions using a region model.

1a.
Sam said that the two squares below have the same fraction of shaded area. Use a drawing and explain why you think Sam is right or wrong.



1b.
Sam said that the two squares below have the same fraction of shaded area. Use a drawing and explain why you think Sam is right or wrong.

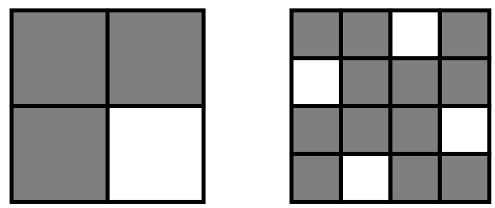


Figure 1. Area Task for Third Grade.

Think carefully about the following question. Write a complete answer. You may use drawings, words, and numbers to explain your answer. Be sure to show all of your work.

a. José ate ½ of a pizza
Ella ate ½ of another pizza

José said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that José could be right.

b. A pizza is sliced into 10 equal pieces and José ate 4 slices of the pizza.
Another pizza is sliced into 5 equal pieces and Ella ate 2 slices of the pizza.

José said that he ate more pizza than Ella, but Ella said they both ate the same amount. Use words and pictures to show that José could be right.

Figure 2. Pizza Task for Fourth Grade.

Data Analysis

A qualitative analysis on the open-ended task questions followed data collection and included open and axial coding (Strauss & Corbin, 1998; Merriam, 2009; Moghaddam, 2006). First, pairs of researchers scored a sample of students' solutions for each problem as correct or incorrect. Incorrect responses were then categorized using open (i.e., descriptive) coding to identify patterns in students' errors. Next, researchers used axial coding to examine the descriptive codes. Researchers grouped similar categories together and identified patterns and relationships among categories. The axial coding resulted in preliminary problem-specific scoring rubrics based on the accuracy of models used in representing the mathematics and how the students used those models in their written solutions.

Next, researchers used the scoring rubrics to independently score and code the entire set of 371 students' open-ended responses. After independent coding, researchers met to compare codes and to discuss discrepancies. In cases of discrepancy, researchers discussed the particular student response and reached a consensus decision. An examination of the error patterns across the entire data set revealed additional patterns, and led researchers to revise the preliminary scoring rubric to distinguish those trends more closely. Consequently, researchers scored and coded the entire data set a second time based on the revised rubrics (see Figure 3). Finally, the frequency and percentage of students in each coding category were tabulated to identify common errors on each of the assessment tasks resulting in the generation of hypotheses about relationships between visual static models of fraction concepts and students' written solutions to problems based on those models.

3a. Rubric for Area Task

Code	Meaning of Code
1	Completely wrong: no attempt or indicated that Sam was wrong
2	Correct spatial drawing OR numerical explanation: either incomplete explanation OR did not provide a drawing as proof
3	Correct spatial drawing AND numerical explanation: explained that four little squares were equivalent to one big square and provided a drawing as proof

3b. Rubric for Pizza Task

Code	Meaning of Code
1	Equivalence: state that the fractions are equivalent without considering size of whole
2	Uneven parts: indicate that the only way José's piece could be larger is if the pieces were not cut evenly
3	Focus on whole number: indicate that José was looking at the whole numbers of the fractions when comparing
4	Correct: explanation or drawing shows a difference in the sizes of the two pizzas

Figure 3. Rubrics for scoring open-ended assessment items: Area and Pizza Tasks.

Results

The overall research question for this study asked: How do visual static models influence students' written solution methods? To generate hypotheses about this relationship, the results for both assessment items are presented. The first section provides descriptive frequencies of errors for each of the assessment tasks. The second section provides descriptive examples of how the given and student-created visual static models related to students' written solutions on the assessment tasks.

Frequencies of Misconceptions

Area task. The first research sub-question asked: What types of misconceptions do students' written solutions commonly reveal on fraction tasks involving either given or student-created visual static models? Third-grade students' written solutions for the Area Task revealed a wide range of conceptual understanding. Table 1 reports students' distribution of responses for 162 third graders on each model type in this task.

Table 1
Distribution of Responses for the Third-Grade Area Task by Model Type

	Area Task Model Type	
Response	Adjacent Squares	Scattered Squares
Correct drawing AND explanation	41 (25.3%)	44 (27.1%)
Correct drawing OR explanation	41 (25.3%)	40 (24.7%)
Completely wrong	80 (49.4%)	78 (48.1%)
Total	162	162

As Table 1 shows, about half of the third-grade students were unsuccessful on this task. Only about one quarter of the students provided both a complete and accurate model and explanation. Responses were scored incomplete if the student agreed that the models were equal but did not provide a comprehensible drawing or explanation. Students' level of accuracy remained virtually the same for the two different models (adjacent and scattered) on this task.

Pizza task. Fourth-grade students' solutions for the Pizza Task revealed a wide range of conceptual understanding that varied according to the model for the task. Table 2 reports students' distribution of responses for 209 fourth graders on each type of model for this task.

Table 2
Distribution of Responses for the Fourth-Grade Pizza Task by Model Type

	Pizza Task Model Type	
Response	1/2 & 1/2	2/5 & 4/10
Correct	43 (20.6%)	10 (4.8%)
Equivalent fractions error	125 (59.8%)	79 (37.8%)
Focus on whole numbers error	20 (9.6%)	54 (25.8%)
Uneven parts error	21 (10.0%)	9 (4.3%)
Not answered; indecipherable	0 (0%)	57 (27.2%)
Total	209	209

As Table 2 shows, the fourth-grade students were more successful with the "½ and ½" model than with the "2/5 and 4/10" model of the Pizza Task (20.6% compared to 4.8%). An examination of student responses revealed three common student errors. First, students most commonly claimed that José was wrong because the fractions were equivalent (i.e., ½ = ½, 2/5 = 4/10). Therefore, they concluded that both children ate the same amount and did not consider that the two wholes could be different sizes. Second, many students argued José's error by comparing the whole numbers of the numerators and denominators. For example, one student partitioned José's pizza into eight sections and suggested that José thought that 4/8 was bigger than ½ because 4 is bigger than 1. This error occurred more often on the "2/5 and 4/10" task (25.8%) than on the "½ and ½" task (9.6%). Finally, some students reasoned that José's portion of the pizza must have been cut slightly larger than the other pieces. Ten percent of the students offered this explanation for the "½ and ½" task, but only 4.3% of the students offered a similar explanation for the "2/5 and 4/10" task. Overall, it seemed that students had difficulty visualizing a model with pizzas of two different sizes.

Relationship Between Visual Static Models and Students' Written Solutions

Area task. The second research sub-question asked: What is the relationship between given or student-created visual static models of fraction concepts and students' written solutions on open-ended problems? Although third-grade students performed similarly on both model types for the Area Task, a close examination of students' work indicates differences in how the visual static models related to students' written solutions. When working with the adjacent squares model of this task, successful students' written explanations more frequently referred to the action of "[moving] the dark square into the empty space" (see Figure 4).

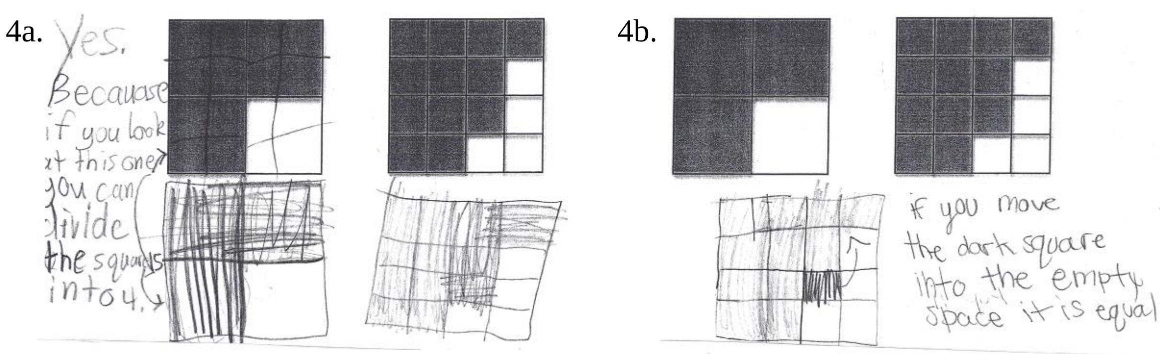


Figure 4. Successful use of the model in the Area Task (adjacent model).

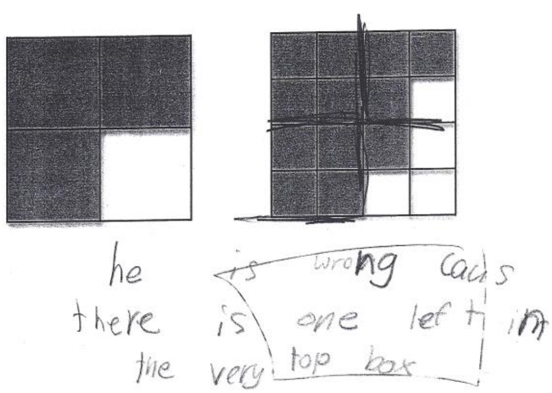


Figure 5. Unsuccessful use of the model in the Area Task (adjacent model).

Unsuccessful students did not provide written information in their responses that indicated that they were able to visualize this action. It is most likely that these students simply perceived the two models as not identical, and therefore concluded that the models did not represent equivalent fractions (see Figure 5).

When working with the scattered squares model for this task, successful students' written explanations more frequently identified four smaller squares as equivalent to one larger square (see Figure 6a). Unsuccessful students typically focused only on the number of squares and not on the size of the squares to determine if the amounts were equivalent (see Figure 6b).

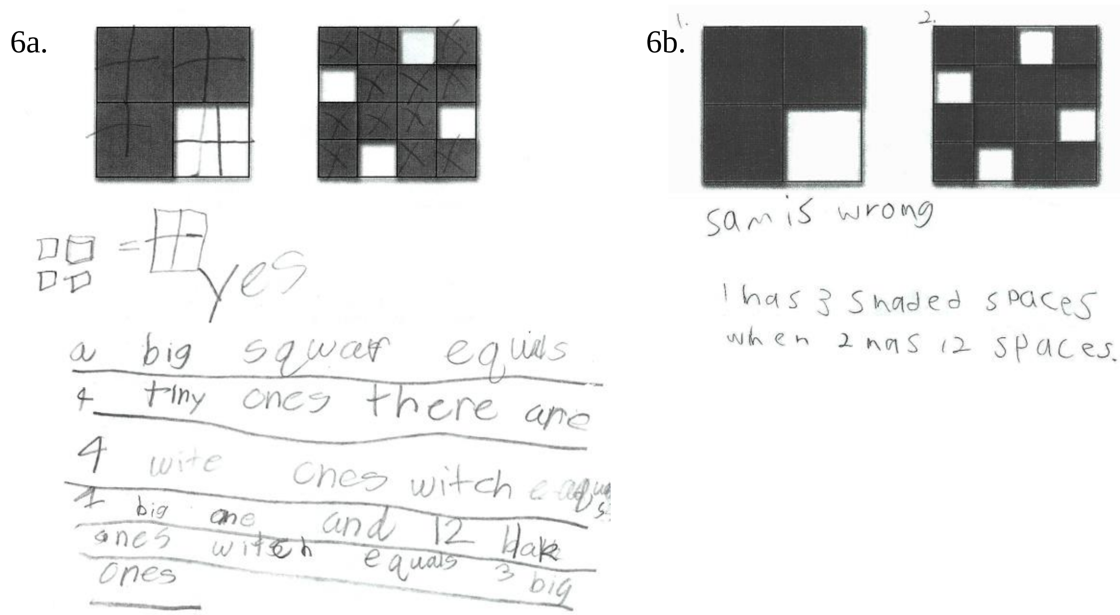


Figure 6. Successful and unsuccessful use of the model in the Area Task (scattered model).

Pizza task. The Pizza Task did not provide students with a visual static model of the problem. Instead, it required the students to develop their own visual static model of the situation (see Figure 7).

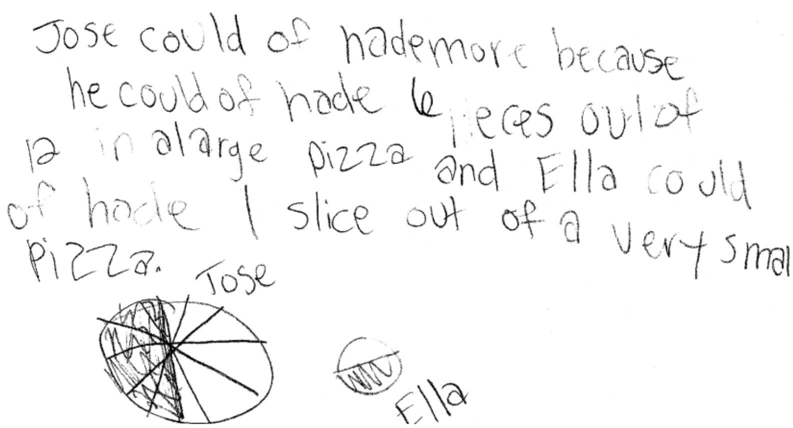
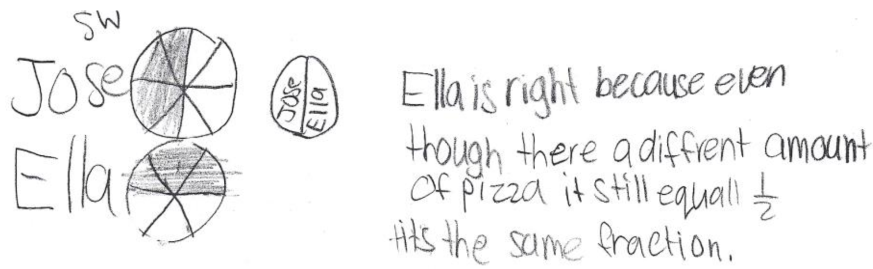


Figure 7. Successful use of a student-generated model in the Pizza Task ("½ and ½" model).

Students' success in solving this task relied heavily on their own visualization skills rather than on their interpretation of a given visual static model. For example, based on their drawings, most students visualized two pizzas of equal size or one pizza cut in half (see Figure 8).

These drawings represented a limited view of the solution possibilities for this task; students did not consider the possibility of different-sized wholes. Even though many students demonstrated proficiency with identifying equivalent fractions, their focus on fraction equivalency prevented them from considering the possibility of different-sized wholes. In the case of the Pizza Task, students' errors may have been caused by a misinterpretation of the problem rather than by a mathematical misconception of fractions.

8a.



8b.

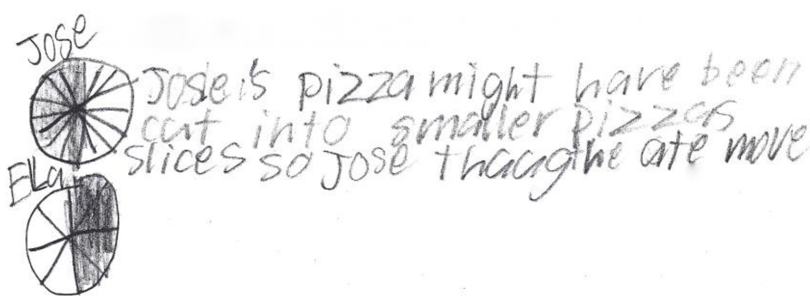


Figure 8. Unsuccessful use of a student-generated model in the Pizza Task ("½ and ½" model).

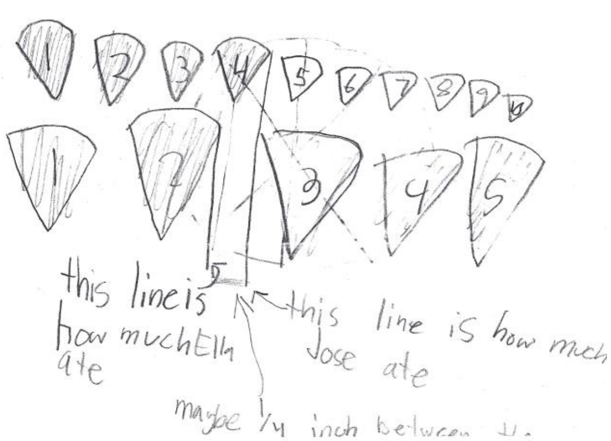


Figure 9. Unsuccessful use of a student-generated model in the Pizza Task ("2/5 and 4/10" model).

At times, the models drawn by the students actually hindered their success with the task. For example, Figure 9 shows a student's drawing of ten smaller pizza slices (José's) lined up with five larger pizza slices (Ella's). The student then drew lines to compare the end of four of José's slices and two of Ella's slices. Unfortunately, even though the student correctly concludes that José ate more pizza, this model does not accurately portray the relationship between the two fractions.

Discussion

This study used a large sample of students' solution models to deconstruct the relationship between visual static models and students' written solutions to fraction problems. Results indicate that common student errors relate to how students interpret the given model or their own model of the problem situation. Results also indicate that students' flexibility with visual static models is related to successful written solutions. These results are discussed in the sections that follow.

What types of misconceptions do students' written solutions commonly reveal on fraction tasks involving either given or student-created visual static models?

This study highlights two misconceptions students have when developing understandings of fraction situations. First, in the Area Task, students unsuccessful with the scattered squares model often focused on the number and not on the size of the squares to determine equivalence. Students' incorrect explanations and answers showed that they were likely considering whole numbers, rather than fractions in making the comparison. This result supports recent research findings that from a young age, number may be more influential than size in children's quantitative problem solving (e.g., Libertus, Starr, & Brannon, 2013).

In the Pizza Task, most students made the assumption that the pizzas were the same size. Because of this, students were unable to generate an accurate model of the mathematical counter-example. A possible explanation of students' difficulty with this problem is that the students did not relate the mathematical pizza context to real-world pizzas that come in many different sizes. This finding is consistent with Verschaffel, Greer, and De Corte's (2007) observation that without a connection to real-world mathematics, students tend to suspend their sense-making and "answer word problems without taking into account realistic considerations about the situations described in the text" (p. 586). Students' difficulty with making sense of the Pizza Task is also consistent with other studies reporting students' tendencies to give an answer to word problems without considering real-world implications of a given situation (Baruk, 1985; Greer, 1993; Verschaffel, et al., 1994).

Another possible explanation of students' difficulty with this problem is that they misinterpreted the task. According to diSessa's (2002) coordination class theory, students attend to what they consider the most important facts of the problem situation and design their solution strategies accordingly. For example, measurement error while cutting pizza slices may be a realistic interpretation. When sharing a pizza, the number of pieces shared is usually more significant than the size of the pieces. This may partially explain some students' reliance on whole number thinking in this situation. Therefore, it can be argued that students' difficulty with the Pizza Task stemmed from a misinterpretation, rather than a misconception (i.e., a wellrehearsed and trusted incorrect idea). Some of the students' interpretations make some sense in the real world even if they do not match the world of mathematics.

An important result to consider is that the two models of the Pizza Task elicited different answers and student misconceptions. In the "½ and ½" model, over half (59.8%) of the students inappropriately used equivalent fractions (i.e., ½ = ½, so the two portions must be equal) to justify their answer. Yet, in the "2/5 and 4/10" model, only 37.8% of the students used the same reasoning. Similarly, students were more likely to consider the numerator and denominator as whole numbers on the "2/5 and 4/10" model than on the "½ and ½" model. This finding suggests that students have a strong concept image (Tall & Vinner, 1981) of fractions such as one half, but not of fractions such as 2/5 or 4/10—possibly because they have had many more experiences seeing the re-representation (Scaife & Rogers, 1996) of ½ and because circular regions are difficult to measure precisely when drawing fraction models. They are less likely to compare or operate on the numerator and denominator of one half (or fractions equivalent to one half) because of this concept image. This pattern reflects the two types of cognitive schema—numerical and global quantitative—described by Moss and Case (1999). These cognitive schemes allow children to understand relative proportions and a semi-abstract understanding of basic fractions such as ½ and ¼. Additionally, reasoning with the 2/5 and 4/10 models involves coordinating the measures and operator sub-constructs (Behr, Lesh, Post, & Silver, 1983) making it more difficult than reasoning with the ½ model. This explains children's ability to work with the ½ model more successfully than the 2/5 and 4/10 models.

What is the relationship between given or student-created visual static models of fraction concepts and students' written solutions on open-ended problems?

The models that students experience, either visually or mentally, relate to students' written solutions to problems. The results from this study shed light on ways that students use these models. First, in the Area Task, most of the students completed the task correctly without dealing directly with fractions because they were able to use the model. Instead of counting the squares to determine equivalent fractions (i.e., using conservation of ratio), they proved equivalence by visualizing the squares moving to different locations. Such movement of squares supports Piaget's (1952) notion of conservation of area. However, relying on the drawn model alone may also divert students' attention away from the numerical relations among the fractional numbers. As long as no parts of the model are deleted or inserted, the model will still represent the same amount regardless of the location of the parts. Some students wrote solutions based on conservation of ratio, but the majority of students in this study relied on conservation of area in their written solutions. The ability to interpret visual static models of fractions in this way may be a precursor to understanding fraction concepts.

Second, in the Pizza Task, students were not given a visual static model on which to base their solutions. Their success with this task depended on their ability to visualize the situation with two different-sized wholes. A large majority of students were unable to satisfactorily complete the two different models of this task (79.4% and 95.2%, respectively). Clearly, self-generation of a representation is more cognitively demanding than working with given representations (Clark, Nguyen, & Sweller, 2006; Woleck, 2001). This finding suggests that students' limited view of fraction models inhibited their success on this task. As noted above, when students fail to visualize mathematical concepts in the real world, they develop a limited conception of the meaning of the mathematics. Students' ability to visualize mathematical concepts is both a process and product of quality experiences in mathematics (Arcavi, 2003). These results also support van Garderen's (2006) findings that well-developed visualization skills contribute greatly to high-achieving students' success in mathematics.

Conclusion

The results of this study indicate that just being presented with visual static models in assessment situations does not guarantee that students will be able to successfully generate their own models or use the given models to accurately solve mathematics problems. The analyses suggest that common misconceptions relate to how a student interprets either the given model or his or her own model of the situation. Differences in students' written solutions could be influenced by instructional strategies or other exposures to mathematical representations including real-world situations. We hypothesize that when students have a solid understanding of real-world mathematical situations, they can successfully create and interpret visual static models to make sense of mathematics. As students manipulate mental objects and consider real-world applications, they actively participate in their own knowledge construction and develop visualization skills. Based on the results of this study, further research is needed to determine the factors influencing students' development of visual static models.

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